Optimal Control of a Global Model of Climate Change with Adaptation and Mitigation*

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Abstract

The Integrated Assessment Model (IAM) has extensively treated the adverse effects of climate change and the appropriate mitigation policy. We extend such a model to include optimal policies for mitigation, adaptation and infrastructure investment studying the dynamics of the transition to a low fossil-fuel economy. We focus on the adverse effects of increase in atmospheric CO_2 concentration on households. Formally, the model gives rise to an optimal control problem of finite horizon consisting of a dynamic system with five-dimensional state vector consisting of stocks of private capital, green capital, public capital, stock of brown energy in the ground, and emissions. Given the numerous challenges to climate change policies the control vector is also five-dimensional. Our solutions are characterized by turnpike property and the optimal policy that accomplishes the objective of keeping the CO_2 levels within bound is characterized by a significant proportion of investment in public capital going to mitigation in the initial periods. When initial levels of CO_2 are high, adaptation efforts also start immediately, but during the initial period, they account for a smaller proportion of government's public investment.

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1 Introduction

The Paris, December 2015, COP 20 agreement on climate change is aiming at reducing the temperature increase to below 2^oC. This implies that effective mitigation policies need to be pursued that not only prevent the CO_2 emission from rising further but to reduce the annual emission substantially. The Paris agreement is detailed in the recent IPCC (2018) report that demonstrates a higher probability of limiting global warming to 1.5 C will only be obtained if the reduction of CO2 net emissions from 2020 to 2040 to zero is achieved and from then on the upper bound of CO_2 is not exceeded anymore.

Since those upper limits create great policy challenges we propose here a modeling strategy that, attempts to answer three questions coming up in this context: First, what are the best strategies to keep the CO_2 emission bounded by a predefined upper bound, and, correspondingly, how can one steer down the CO_2 emission if it already has reached too high a level. Second, how can climate policies be scaled up and what resources should be allocated to mitigation and adaptation efforts, especially for the latter, in particular, when climate risk, due to a lack of emission reduction, is rising and future economic, social, and ecological damages can be expected. A third issue is of how the efforts of mitigation and adaptation are funded and how the funds should dynamically be allocated between traditional infrastructure investment, mitigation and adaptation efforts – and in what sequence.

Since mitigation policy means phasing in of renewable energy we also explore what amount of traditional fossil energy is allowed to be extracted when setting some CO_2 emission and temperature constraints. Our dynamic model, as it includes the phasing in of renewable energy along with the issues mentioned above, can be considered an extension of the Integrated Assessment Model(IAM).[1]

We present a dynamic global model with feedback control, representing an optimal control, that allows us to consider the specific policies of infrastructure investment, mitigation and adaptation. The model is micro-founded in the sense that we employ a production technology which uses (private) physical capital and energy as inputs. Labor input is suppressed for simplicity as it is supplied inelastically. There are two sources of energy: non-renewable, brown energy produced by an extractive resource sector and renewable, green energy produced with (private physical) green capital. The emissions from brown energy use are a source of negative externality that directly enters the (instantaneous) felicity function.

In our model the government levies lump-sum taxes to raise revenues,[2] a portion of which provides direct utility, another portion is invested in public (physical) capital, and remaining part is administrative expense. The traditional use of public capital is

[1]For details of the IAM, see Nordhaus et al. (2000) and Nordhaus (2008).

[2]For models with other sources of finance, such as for example bond financing, see Bonen et al (2016), and Orlov et al (2018)

to serve as infrastructure investment that augments the productivity of the production process.[3] In our setup, the government can also use public capital for adaptation and for mitigation and chooses the split between these three competing uses optimally.

The model gives rise to an optimal control problem of finite horizon consisting of a dynamic system with five-dimensional state vector consisting of the stocks of private capital, green capital, public capital, stock of brown energy in the ground, and emissions. The control vector is also a five-dimensional, excluding the choice of split for public capital mentioned earlier.

We characterize the optimal tax and investment policies for the government and examine the resultant paths of important macroeconomic variables, particularly, of those related to energy, emission, and resource extraction. The complexity of the problem, however, necessitates both an analytical approach as well the use of numerical methods.

Solving such a model of finite horizon poses the challenge to show that the turnpike properties are not violated and the trajectories of the finite horizon model can approximate the solution of the infinite horizon case. This in fact can be shown when we present the numerical solutions of the variants of our model with different initial conditions. These numerical solutions allow us to investigate the optimal sequence of climate policy decisions with respect to infrastructure, mitigation and adaptation. In all cases considered in the paper, numerical solutions from our finite-horizon set up recover the turnpike property that is characteristics of the infinite horizon models and hence, good approximation to those models.

In terms of climate policy response, in our model, we find that the optimal policy can keep the CO_2 emission bounded for a wide-range of initial conditions of capital stocks and CO_2 levels. Specifically, we consider scenarios with high levels of capital stocks and high levels of CO_2 and with small levels of capital stocks with both high and low levels of CO_2. In all these cases, we find that the optimal policy that accomplishes the objective of keeping the CO_2 levels within bound is characterized by a significant proportion of investment in public capital going to mitigation in the initial period. In fact, the time path and the proportion going to mitigation are very similar across these scenarios. In cases with high initial levels of CO_2, adaptation efforts also start immediately, but during the initial period, they account for smaller proportion of government's public investment than mitigation and do not change much during the initial period.

The remaining part of the paper is organized as follows. Section 2 describes the optimal control model of climate change. In Section 3, we discuss the necessary conditions for the optimal solution and solve for the stationary solutions for the model. Section 4 works through the details of our numerical methods and describes the main results of the paper. In particular, optimal policy for various initial conditions is discussed.

[3]This infrastructure investment can be considered to representing traditional as well as climate-related infrastructure.

Section 5 concludes.

2 Optimal Control Model of Climate Change

We extend the IAM to include the adverse effects of climate change with a view to study the optimal policies for mitigation of and adaptation to climate change until a transition to fossil-fuel-free green energy infrastructure is successful. The green energy capital is a perfect substitute for fossil fuel in production. The climate change is modeled as an adverse effect of increase in atmospheric CO_2 concentration (M) on utility. The mitigation efforts reduces the proportion of carbon in fossil fuel burned that escapes into the atmosphere as CO_2. In contrast, adaptation alleviates the harmful effects of higher atmospheric CO_2 levels.

The time (t) is continuous, and the horizon is finite (T). The government raises revenue (e_P) which is used for direct, utility-enhancing services and provision of public (physical) capital/infrastructure (G), with possibility of some wastage. To analyze the issue of climate change, besides its traditional use for enhancing productive efficiency in the economy, we allow government to use public capital for mitigation and adaptation.

The output of the production process is given by

$$Y = (\nu_1 G)^\beta A(A_g K_g + A_u u)^\alpha (K_p)^\zeta. \tag{1}$$

$A, A_g, A_u > 0, \alpha, \beta, \zeta > 0$, and $\alpha + \beta + \zeta < 1$. K_g is the stock of green capital, K_p is the stock of (private) physical capital, and $\nu_1 \in (0, 1]$, as mentioned above, is the fraction of public capital (G) used for the traditional purpose of enhancing productive efficiency. Finally, u is the amount of fossil fuel resource extracted and used, measured in terms of its carbon (CO_2) content.

The felicity (utility) function depends on four input components: (i) per-capita consumption C; (ii) the per-capita amount of tax revenue ($\alpha_2 e_P$, $\alpha_2 \in [0, 1]$) used for direct welfare enhancement (e.g., healthcare); (iii) atmospheric concentration of CO_2 (M) above the long-run sustainable level-industrial level; and (iv) the per-capita amount of public capital expenditure ($\nu_2 G$, $\nu_2 \in [0, 1)$) allocated to climate change adaptation.

The optimal control model so defined then has five state variables:

K_p : private physical capital per capita,
K_g : private green capital per capita,
G : public capital per capita,
M : CO_2 (GHG) concentration in the atmosphere,
R : non-renewable resource (fossil energy),

and the five basic control variables are

i_p : investment in physical capital,
i_g : investment in green capital,
e_p : government's net tax revenue,
u : extraction rate from the non-renewable resource,
C : per capita consumption,

In addition, we consider the following three allocations of public capital as control functions:

$\nu_1(t)$: standard infrastructure,
$\nu_2(t)$: adaptation,
$\nu_3(t)$: mitigation.

We denote the state and control variables by

$$X = (K_p, K_g, G, R, M) \in \mathbb{R}^5, \quad U = (i_p, i_g, e_P, u, C) \in \mathbb{R}^5, \quad \nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3.$$

The dynamic system of the global model of climate change is given by

$$\dot{K}_p = i_p - (\delta_p + n)K_p, \tag{2}$$

$$\dot{K}_g = i_g - (\delta_g + n)K_g, \tag{3}$$

$$\dot{G} = \alpha_1 e_P - (\delta_G + n)G, \tag{4}$$

$$\dot{M} = \gamma\, u - \mu(M - \kappa\widetilde{M}) - \theta(\nu_3 \cdot G)^{\phi}, \tag{5}$$

$$\dot{R} = -u, \tag{6}$$

with initial conditions:

$$X(0) = X_0 \tag{7}$$

that will be specified later. The control constraint for the extraction rate u is given by

$$0 \leq u(t) \leq u_{max} \quad \forall\, t \in [0, T]. \tag{8}$$

There are three potential uses of government revenues as mentioned earlier. The amount $\alpha_1 e_P$ is invested in public capital, $\alpha_2 e_P$ provides direct utility, and $(1 - \alpha_1 - \alpha_2)e_P$ is administrative expense/waste. Of the total public capital, G, a fraction ν_1 is the usual/traditional public capital that augments the productivity of the production process. Another fraction ν_2 is used for adaptation. The remaining fraction ν_3 is used for mitigation. Hence, the infrastructural and climate oriented allocations of public capital satisfy the constraints:

$$\nu_k(t) \geq 0, \quad \nu_1(t) + \nu_2(t) + \nu_3(t) = 1 \quad \forall\, t \in [0, T]. \tag{9}$$

Moreover, we have the *resource constraint* which is a control-state equality constraint:

$$\begin{aligned} s(X, U, \nu) \;:=\; & Y - C - i_p - i_g - e_P - u\psi R^{-\tau} \\ & - \tfrac{\chi_p}{2}\left(\tfrac{i_p}{K_p} - \delta_p - n\right)^2 K_p - \tfrac{\chi_g}{2}\left(\tfrac{i_p}{K_g} - \delta_G - n\right)^2 K_g = 0. \end{aligned} \tag{10}$$

Let us now introduce the welfare functional. Recall, the felicity (utility) function depends on (i) per-capita consumption C; (ii) the per-capita tax revenue ($\nu_2 e_P$, $\alpha_2 \in [0,1]$); (iii) atmospheric concentration of CO_2 (M) above the long-run sustainable level-industrial level; and (iv) the per-capita expenditure on adaptation ($\nu_2 G$, $\nu_2 \in [0,1)$). The preferences of the representative household (or the planner) are

$$\int_0^T e^{-(\rho-n)t}\frac{1}{1-\sigma}\left\{\left[C\left(\alpha_2 e_P\right)^{\eta}\left(1-\exp\left(-\xi\left(\nu_2 G\right)^{\omega}\right)\frac{M-\kappa\tilde{M}}{\bar{M}-\kappa\tilde{M}}\right)^{\varepsilon}\right]^{1-\sigma}-1\right\}dt, \tag{11}$$

where $\tilde{M}$ is the preindustrial level of atmospheric CO_2 and $\bar{M}$ is the catastrophic level, with $\kappa\tilde{M}$ being the level that would not need any adaptation and is the long-run sustainable level. $\rho > 0$ is the time rate of preference, $n > 0$ is the rate of population growth, $\sigma > 0$ is the inverse of the elasticity of intertemporal substitution and $\eta \in [0,1]$, $\varepsilon \in [0,1]$, $\xi > 0$, $\omega \in [0,1]$, and $\kappa > 0$ are other parameters. The restrictions on parameters ensure that social expenditures and adaptation are utility enhancing with diminishing marginal utility and carbon emission that increase M reduce utility with increasing marginal disutility.[4]

This approach differs from other models that map emissions to temperature changes and then to reduced productivity-*cum*-output, see Nordhaus et al. (2000). We believe the direct disutility approach better captures the wide ranging impacts of climate change that may include health impacts, ecological loss and heightened uncertainty, in addition to reduced productivity. Finally, note that the discount factor adjusts for the population growth rate n from the pure discount rate ρ as all values are normalized by the population.

The optimal control problem now consists in *maximizing* the welfare functional (11) subject to the dynamical constraint and control constraints. To obtain a more compact form of the optimal control problem we use the vector of state and control variables (X, U, ν) introduced above to write the dynamical system (2)–(5) in the form

$$\dot{X}(t) = f(X(t), U(t), \nu(t)), \quad X(0) = X_0. \tag{12}$$

Furthermore, let us denote the integrand of the welfare functional by

$$f_0(X,U,\nu) = \frac{1}{1-\sigma}\left\{\left[C\left(\alpha_2 e_P\right)^{\eta}\left(1-\exp\left(-\xi\left(\nu_2 G\right)^{\omega}\right)\frac{M-\kappa\tilde{M}}{\bar{M}-\kappa\tilde{M}}\right)^{\varepsilon}\right]^{1-\sigma}-1\right\}. \tag{13}$$

Then the optimal control problem can be written in compact form as follows:

$$\text{Maximize}_{\,U,\nu}\, W(U,\nu = \int_0^T e^{-(\rho-n)t} f_0(X(t),U(t),\nu(t))\,dt \tag{14}$$

subject to the dynamical constraint (12), the control constraints (8), (9) and the mixed control-state constraint (10).

[4]For $\sigma \geq 1$, we only need $\eta, \varepsilon > 0$.

Parameter	Value	Definition
ρ	0.03	Pure discount rate
n	0.015	Population Growth Rate
η	0.1	Elasticity of transfers and public spending in utility
ϵ	1.1	Elasticity of CO_2-eq concentration in (dis)utility
ω	0.05	Elasticity of public capital used for adaptation in utility
σ	1.1	Intertemporal elasticity of instantaneous utility
A	$\in [1, 10]$	Total factor productivity
A_g	$\in [\,1\,,\,5\,]$	Efficiency index of green capital
A_u	$\in [100, 400]$	Efficiency index of the non-renewable resource
α	0.1	Output elasticity of inputs, $A_g K_g + A_u u$
β	0.5	Output elasticity of public infrastructure, $\nu_1 G$
ψ	1	Scaling factor in marginal cost of resource extraction
τ	2	Exponential factor in marginal cost of resource extraction
δ_p	0.1	Depreciation rate of physical capital
δ_g	0.05	Depreciation rate of private capital
δ_G	0.05	Depreciation rate of public capital
χ_p	$\frac{1}{(\delta_p+n)\Omega_p}$	
χ_g	$\frac{1}{(\delta_g+n)\Omega_g}$	
Ω_p	$\in [5, 15]$	
Ω_g	$\in [5, 15]$	
α_1	0.2	Proportion of tax revenue allocated to new public capital
α_2	0.5	Proportion of tax revenue allocated to transfers and public consumption
$\bar{r}$	0.07	World interest rate (paid on public debt)
$\widetilde{M}$	2.5	equilibrium concentration
γ	0.9	Fraction of greenhouse gas emissions not absorbed by the ocean
μ	0.01	Decay rate of greenhouse gases in atmosphere
κ	1	Atmospheric concentration stabilization ratio (relative to $\widetilde{M}$)
θ	0.01	Effectiveness of mitigation measures
ϕ	$\in [\,0.2,\,1\,]$	exponent in mitigation term $(\nu_3\, g)^{\phi}$

Table 1: Parameter values

3 Necessary Conditions and Stationary Solutions

We formulate the necessary conditions of the Maximum Principle (cf. Hestenes [9], Pontryagin et al. [16], Hartl et al. [8]), Maurer et al. [2016], Greiner et al [2010]) using the current-value Hamiltonian

$$H(X, \lambda, U, \nu) = f_0(X, U, \nu) + \lambda\, f(X, U, \nu), \tag{15}$$

where

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (\lambda_{Kp}, \lambda_{Kg}, \lambda_G, \lambda_M, \lambda_R)$$

denotes the adjoint variables.

The adjoint variables $\lambda(t)$ of the current-value Hamiltonian are related to the adjoint variables $\tilde{\lambda}(t)$ of the standard Hamiltonian by

$$\lambda(t) = e^{(\rho - n)}\,\tilde{\lambda}(t),$$

which leads to the modified adjoint equation below. Since the process is subject to the mixed control-state constraint (10), we consider the augmented current-value Hamiltonian

$$\mathcal{H}(X, \lambda, \eta, U, \nu) = H(X, \lambda, U, \nu) + \eta\, s(X, U, \nu). \tag{16}$$

Let (X, U, ν) be an optimal solution of the control problem of maximizing (14) subject to the constraints (12),(8),(9), (1), and (10). To formulate the minimum conditions for the controls we introduce the control set at time t:

$$\begin{aligned} \Omega(t) \;=\; & \{\, (U, \nu) \in \mathbb{R}^8 \,|\, i_p(t), i_g(t), e_P(t) \geq 0,\; C(t) > 0,\; 0 \leq u(t) \leq u_{max}, \\ & \nu_k \geq 0 \;(k = 1, 2, 3),\;\; \nu_1(t) + \nu_2(t) + \nu_3(t) = 1, s(X(t), U, \nu) = 0 \,\} \end{aligned} \tag{17}$$

Then the Maximum Principle asserts the existence of a piecewise continuous adjoint function $\lambda : [0, t_f] \to \mathbb{R}^5$ and a continuous multiplier function $\eta : [0, t_f] \to \mathbb{R}$ such that the following conditions hold

Adjoint equations:

$$\begin{aligned} \dot{\lambda}(t) &= (\rho - n)\lambda(t) - \tfrac{\partial \mathcal{H}}{\partial X}(X(t), \lambda(t), \eta(t), U(t), \nu(t)) \\ &= (\rho - n)\lambda(t) - \tfrac{\partial H}{\partial X}(X(t), \lambda(t), \eta(t), U(t), \nu(t)) - \eta(t)\tfrac{\partial s}{\partial X}(X(t), U(t), \nu(t)). \end{aligned} \tag{18}$$

Minimum conditions for controls

$$H(X(t), \lambda(t), U(t), \nu(t)) \;=\; \max{}_{(U,\nu)\in\Omega(t)}\; H(X(t), U, \nu) \tag{19}$$

Local minimum conditions when $i_p(t), i_g(t), e_P(t), C(t) > 0$:

$$\frac{\partial \mathcal{H}}{\partial v}(X(t), \lambda(t), U(t), \nu(t)) = 0 \quad \text{for} \quad v \in \{\, i_p, i_g, e_P, C \,\} \tag{20}$$

We shall not further analyse the minimum conditions for the allocations $\nu(t)$.

Now let us study the stationary solution of the canonical system and the minimum conditions. Since our computations show that $\nu(t_f) \approx (1,0,0)$, we fix $\nu = (1,0,0)$ in the following analysis of the stationary point. In view of the equation $\dot{R} = u$ it is clear that the extraction rate is $u = 0$. Hence, we can discard the adjoint equation λ_R in the following analysis. Therefore, we consider the following 13 equations for the variables

$$(K_p, K_g, G, M),\ (\lambda_{Kp}, \lambda_{Kg}, \lambda_G, \lambda_M,\ (i_p, i_g, e_P, C),\ \eta.$$

Equations for stationary state:

$$\begin{aligned} 0 &= i_p - (\delta + n)K_p, \\ 0 &= i_g - (\delta + n)K_g, \\ 0 &= \alpha_1 e_P - (\delta_G + n)G, \\ 0 &= 0 - \mu(M - \kappa \tilde{M}). \end{aligned} \tag{21}$$

Equations for adjoint (costate) variable:

$$\begin{aligned} 0 &= (\rho - n)\lambda_{Kp} - \tfrac{\partial \mathcal{H}}{\partial K_p} \\ &= (\rho - n)\lambda_{Kp} + \lambda_{Kp}(\delta_p + n) - \eta(\chi_p(i_p/K_p - \delta_p - n)(-i_p/K_p) \\ &\quad +0.5\chi_p(i_p/Kp - \delta_p - n)^2 - \tfrac{\partial Y}{\partial K_p} \\ 0 &= (\rho - n)\lambda_{Kg} - \tfrac{\partial \mathcal{H}}{\partial K_g} \\ &= (\rho - n)\lambda_{Kg} + \lambda_{Kg}(\delta_g + n) - \eta(\chi_g(i_g/K_g - \delta_g - n)(-i_g/K_g) \\ &\quad +0.5\chi_g(i_g/Kg - \delta_g - n)^2 - \tfrac{\partial Y}{\partial K_g} \\ 0 &= (\rho - n)\lambda_G - \tfrac{\partial \mathcal{H}}{\partial G} \\ &= (\rho - n)\lambda_G + \lambda_G(\delta_G + n) - \tfrac{\partial Y}{\partial G} - \tfrac{\partial f_0}{\partial G} \\ 0 &= (\rho - n)\lambda_M - \tfrac{\partial \mathcal{H}}{\partial M} = (\rho - n)\lambda_M + \eta\lambda_M(\delta_G + n) - \tfrac{\partial f_0}{\partial M}. \end{aligned} \tag{22}$$

Equations for maximizing controls $v \in \{i_p, i_g, e_P, C\}$

$$\begin{aligned} 0 &= \tfrac{\partial \mathcal{H}}{\partial i_p} = \lambda_{Kp} + \eta(1 + \chi_p(i_p/K_p - \delta_p - n)), \\ 0 &= \tfrac{\partial \mathcal{H}}{\partial i_g} = \lambda_{Kg} + \eta(1 + \chi_g(i_g/K_g - \delta_g - n)), \\ 0 &= \tfrac{\partial \mathcal{H}}{\partial e_P} = \lambda_G \alpha_1 + \eta + \tfrac{\partial f_0}{\partial e_P}, \\ 0 &= \tfrac{\partial \mathcal{H}}{\partial C} = \tfrac{\partial f_0}{\partial C} + \eta. \end{aligned} \tag{23}$$

Finally, we consider the equation resulting from the mixed control-state (10):

$$\begin{aligned} 0 &= s(X, U, \nu) = Y - C - i_p - i_g - e_P - u\psi R^{-\tau} \\ &\quad - \tfrac{\chi_p}{2}\left(\tfrac{i_p}{K_p} - \delta_p - n\right)^2 K_p - \tfrac{\chi_g}{2}\left(\tfrac{i_p}{K_g} - \delta_G - n\right)^2 K_g\,. \end{aligned} \tag{24}$$

The partial derivatives of the integrand $f_0(X, U, \nu)$ in equations (21) -(23) are given by

$$\begin{aligned} \tfrac{\partial f_0}{\partial C} &= \{(1-\sigma)C^{-\sigma}(\alpha_2 e_P)^{\eta(1-\sigma)}[1 - \exp(-\xi(\nu_2 G)^\omega) \\ &\quad \cdot(M - \kappa\tilde{M})/(\bar{M} - \kappa\tilde{M})]^{\epsilon(1-\sigma)} - 1\}/(1-\sigma), \\ \tfrac{\partial f_0}{\partial e_P} &= \{C^{1-\sigma}\alpha_2\eta(1-\sigma)(\alpha_2 e_P)^{\eta(1-\sigma)-1}[1 - \exp(-\xi(\nu_2 G)^\omega) \\ &\quad \cdot(M - \kappa\tilde{M})/(\bar{M} - \kappa\tilde{M})]^{\epsilon(1-\sigma)} - 1\}/(1-\sigma), \\ \tfrac{\partial f_0}{\partial G} &= 0 \quad \text{in view of } \nu_2 = 0 \text{ in the steady state.} \end{aligned} \tag{25}$$

The partial derivatives of the production function $Y(X, U, \nu)$ in equation (1) are given by

$$\begin{aligned} \frac{\partial Y}{\partial K_p} &= A(\nu_1 G)^{\beta}(A_g K_g + A_u u)^{\alpha} \zeta K_p^{\zeta-1}\,, \\ \frac{\partial Y}{\partial K_g} &= A(\nu_1 G)^{\beta} \alpha A_g (A_g K_g + A_u u)^{\alpha-1} K_p^{\zeta}\,, \\ \frac{\partial Y}{\partial G} &= A\beta\nu_1(\nu_1 G)^{\beta-1}(A_g K_g + A_u u)^{\alpha} K_p^{\zeta}\,, \end{aligned} \tag{26}$$

We use AMPL and Ipopt (described in Section 4) to solve the 13 equations in (21)–(24) with parameters in Table 1 and obtain the *stationary solution*:

$$\begin{array}{lllll} K_p = 2.2164, & K_g = 0.53731, & G = 0.66746, & M = 3.0, & \\ i_p = 0.25489, & i_g = 0.034925, & e_P = 0.14462, & C = 0.74766, & u = 0, \\ \lambda_{Kp} = 2.24939, & \lambda_{Kg} = 2.24939, & \lambda_G = 3.6216, & \lambda_M = -12.121, & \mu_s = -2.24939. \end{array} \tag{27}$$

Our computations in the next section will demonstrate that the solution shows a *turnpike behavior*, i.e., the state and control trajectories stay close to the stationary values in (27) on a rather large intermediate time interval, once dynamics dictated by initial conditions have played out. However, for a free terminal state the trajectories for the K_p, K_g, G sharply decrease on the terminal time interval. To counteract this behavior we shall impose the stationary values of the state variables as terminal state constraints. This will furnish a good approximation of the infinite-horizon solution.

4 Numerical Solutions

4.1 Discretization and Nonlinear Programming Methods

We choose the numerical approach "First Discretize then Optimize" to solve the optimal control problem $OC(p)$ defined in (12)–(14). The discretization of the control problem on a fine grid leads to a large-scale nonlinear programming problem (NLP) that can be conveniently formulated with the help of the Applied Modeling Programming Language (AMPL) [6]. AMPL can be linked to several powerful optimization solvers. We use the Interior-Point optimization solver IPOPT developed by Wächter and Biegler [17]. The details of the discretization methods may be found in [1, 4, 11]. The subsequent computations for the terminal time $T = 200$ are performed with $N = 1000$ to $N = 5000$ grid points using the trapezoidal rule as integration method. Choosing the error tolerance $tol = 10^{-8}$ in IPOPT, we can expect that the state variables are correct up to 6 or 7 decimal digits. The Lagrange multipliers and adjoint variables can computed a posteriori in IPOPT thus enabling us to verify the necessary optimality conditions.

4.2 Solutions with Free Terminal States

We begin our numerical analysis with an illustrative example with free terminal state. We choose the following as the terminal time and the initial conditions:

$$T = 200: \quad K_p(0) = 2.5,\ K_g(0) = 0.3,\ G(0) = 0.8,\ M(0) = 3.25, ; R(0) = 1.$$

As integration method we choose the trapezoidal rule with $N = 2000$ grid points. The control and state trajectories are displayed in Figure 1.

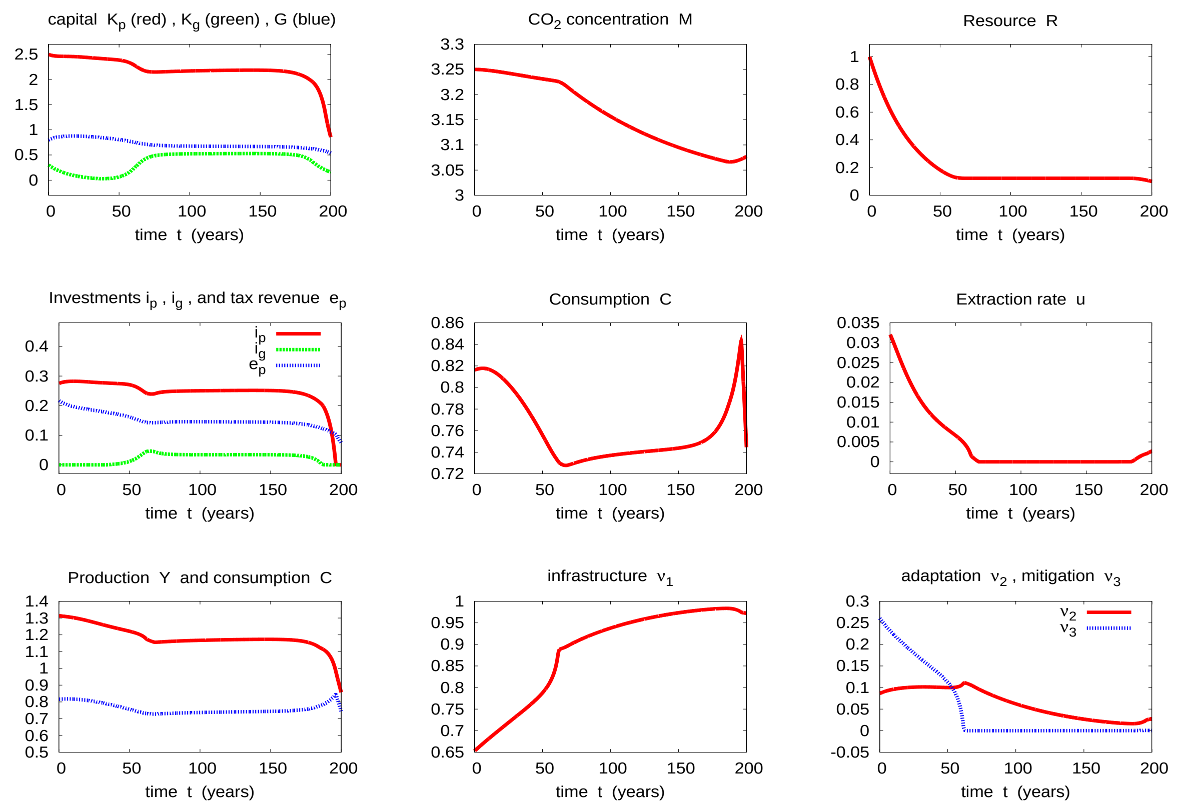

Figure 1: State and control trajectories for $T = 200, K_p(0) = 2.5, K_g(0) = 0.3, G(0) = 0.8, M(0) = 3.25, R(0) = 1$, and free terminal state $X(t_f)$. *Top row*: (left) physical capital K_p, green capital K_g and government capital G, (middle) CO_2 concentration M, (right) resource R. *Middle row*: (left) investments i_p, i_g and tax revenue e_P, (middle) consumption C (right) extraction rate u. *Bottom row*: (left) consumption C and productivity Y, (middle) infrastructure ν_1, (right) adaptation ν_2 and mitigation ν_3.

As Figure 1 shows, even for large initial conditions and high atmospheric CO_2 concentration, and free terminal states of the variables, the policy variables are able to steer down the CO_2 concentration toward the steady state. This of course holds only if there is no bifurcation and regime switch to a worse steady state, as studied in Greiner et al. (2010). In fact, in our model here, one can observe that all state variables and control variables move to some reasonable steady state that has the CO_2

concentration under control. This is a reflection of the turnpike property that characterizes dynamic macroeconomic models. Once the non-renewable fossil fuel extraction becomes economically unproductive (with $u(t) \approx 0$) the model converges to the turnpike corresponding to the long-run steady state solution described in Section 3, until unnecessary dynamics introduced by finite terminal time kicks in later. To eliminate this unwanted terminal dynamics, in our further analysis, we choose the terminal states to be the same as the steady state solution when we solve the model.

Note that infrastructure policy first gives way to expanding mitigation efforts which is large at the initial periods and the adaptation policy kicks in later. Note also that the ratios of our macroeconomic variables stay over time in a reasonable range. This holds for all of our subsequent simulations.

4.3 Solutions with Prescribed Terminal States

Our analysis in this section begins with the consideration of high initial value of the emission stock. We examine what optimal policy looks like in this situation. We investigate cases with both a small and a high value of various capital stocks. Thereafter, we also look for the situation with low initial value of the emission stock.

4.3.1 Small Initial Capital Stocks and High Emission Stock

Let us first consider small initial values for capital stocks[5] and, as stated earlier, prescribe as terminal conditions for the state variables K_p, K_g, G the stationary values (27):

$$
\begin{array}{lllll}
K_p(0) = 1, & K_g(0) = 0.02, & G(0) = 0.2, & M(0) = 3.25, & R(0) = 1, \\
K_p(t_f) = 2.2164, & K_g(t_f) = 0.53731, & G(t_f) = 0.66746. & &
\end{array}
$$

Figure 2 shows that for small initial capital stocks and high atmospheric CO_2 concentration, the policy choices are able to steer down the CO_2 concentration toward the steady state, after some increase during the initial periods. Here too, we can observe that all state and control variables move to some reasonable steady state that keeps the CO_2 concentration under control. In particular, output, consumption, and investments all remain fairly stable, except for brief initial transition associated with low initial levels of capital stocks. Figure 3 demonstrates that there are corresponding properties holding for the trajectories of the adjoint variables.

Note that, here again, the spending on traditional infrastructure is lower at the beginning with large efforts at mitigation as well as rising efforts at adaptation. Overall, mitigation is the focus of the public investment policy in the beginning as it accounts for more than twice of investment than adaptation. Over time, the mitigation policy is

[5] This may correspond to developing economies with low per capita income, yet facing already a high atmospheric CO_2 concentration.

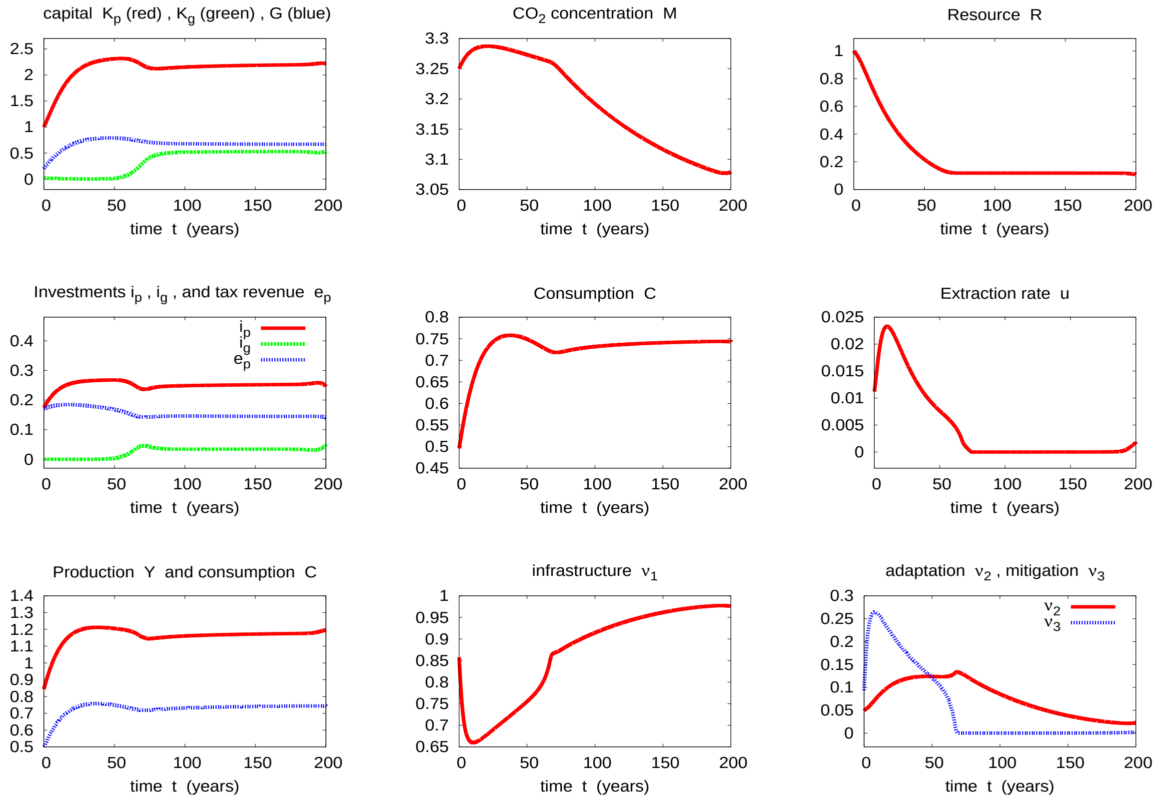

Figure 2: State and control trajectories for $T = 200$, "small" initial values $K_p(0) = 1, K_g(0) = 0.02, G(0) = 0.2, R(0) = 1, M(0) = 3.25$ and prescribed terminal stationary values $K_p(t_f) = 2.2164, K_g(t_f) = 0.53731, G(t_f) = 0.66746$ in (27). *Top row*: (left) physical capital K_p, green capital K_g and government capital G, (middle) CO_2 concentration M , (right) resource R. *Middle row*: (left) investments i_p, i_g and tax revenue e_P, (middle) consumption C (right) extraction rate u. *Bottom row*: (left) consumption C and productivity Y, (middle) infrastructure ν_1, (right) adaptation ν_2 and mitigation ν_3.

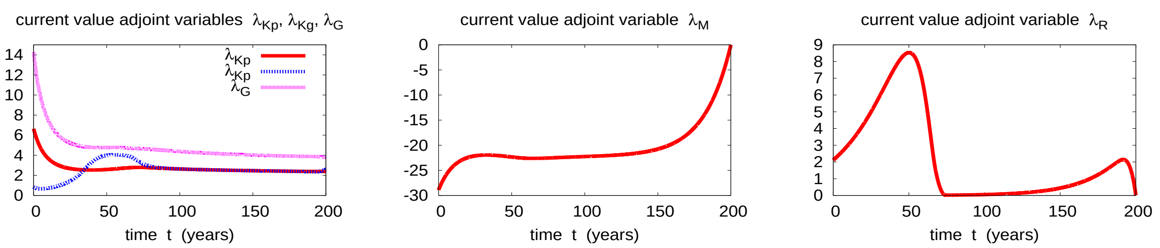

Figure 3: Adjoint variables for "small" initial values. (left) current value adjoint variables $\lambda_{Kp}, \lambda_{Kg}, \lambda_G$, (middle) current value adjoint variable λ_M, (right) current value adjoint variable λ_R.

first to slow down, but adaptation continues for a long time. During the initial periods, the spending on mitigation and adaptation together starts with accounting for 35% of the public investment in infrastructure, gradually falling to 15%, before converging asymptotically to zero over time. The optimal solution also leaves some of the fossil fuels (R) in ground.

4.3.2 Large Initial Capital Stocks and High Emission Stock

We now consider a complementary scenario in which initial values for capital stocks, $K_p(0)$ and $G(0)$, exceed the stationary values, and, again, impose the stationary values of K_p, K_g, G as the terminal constraints:

$$\begin{array}{lllll} K_p(0) = 3, & K_g(0) = 0.5, & G(0) = 1.0, & M(0) = 3.25, & R(0) = 1, \\ K_p(t_f) = 2.2164, & K_g(t_f) = 0.53731, & G(t_f) = 0.66746. & & \end{array}$$

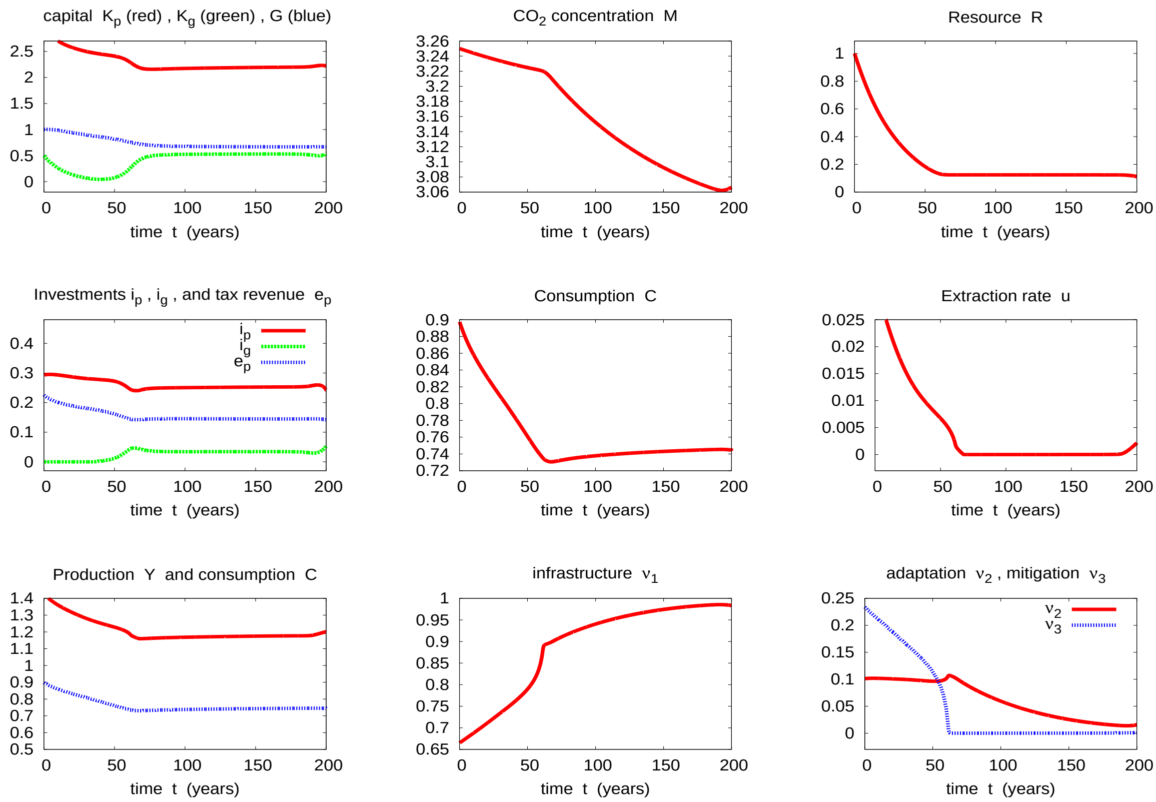

Figure 4: State and control trajectories for "large" initial values $K_p(0) = 3, K_g(0) = 0.5, G(0) = 1.0, M(0) = 3.25, R(0) = 1$ and terminal constraints $K_p(t_f) = 2.2164, K_g(t_f) = 0.53731, G(t_f) = 0.66746$. *Top row*: (left) physical capital K_p, green capital K_g and government capital G, (middle) CO_2 concentration M, (right) resource R. *Middle row*: (left) investments i_p, i_g and tax revenue e_P, (middle) consumption C (right) extraction rate u. *Bottom row*: (left) consumption C and productivity Y, (middle) infrastructure ν_1, (right) adaptation ν_2 and mitigation ν_3.

Figure 4 shows that even with large initial capital stocks and high atmospheric CO_2 concentration, the policy variables are able to steer down the CO_2 concentration toward the steady state. This time, however, without any increase during the initial periods. Again, we can observe that all state and control variables (and adjoint variables) move

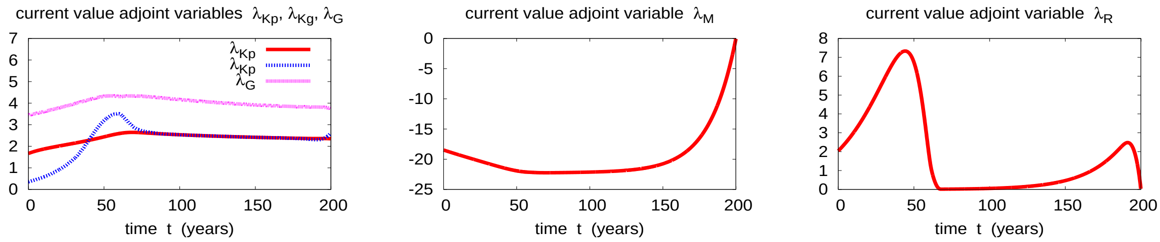

Figure 5: Adjoint variables for "small" initial values. (left) current value adjoint variables $\lambda_{Kp}, \lambda_{Kg}, \lambda_G$, (middle) current value adjoint variable λ_M, (right) current value adjoint variable λ_R.

to some reasonable steady state that keeps the CO_2 concentration under control. In particular, output, consumption, and investments all remain fairly stable, after the initial periods. But, as in this case, the economy has excess capital stocks, it reduces them slowly during the initial periods, which imparts a discernible (downward) slope to consumption, output, and capital stock for the entire duration of the initial periods. Figure 5 shows again the corresponding trajectories of the adjoint variables.

As in the case with low initial capital stocks, the spending on traditional infrastructure is lower at the beginning with large efforts at mitigation as well as rising efforts at adaptation. Once again, mitigation is the focus of the public investment policy in the beginning as it accounts for more than twice of investment than adaptation. Over time, the mitigation policy slows down, but adaptation continues for a long time. Once again, during the initial periods, the spending on mitigation and adaptation together starts with accounting for 35% of the public investment in infrastructure, gradually falling to 10%, before converging asymptotically to zero over time. The optimal solution again leaves some of the fossil fuels (R) in the ground.

4.3.3 Small Initial Capital Stocks and Low Emission Stock

We next turn to the case with small initial values for the capital stocks and low CO_2 concentration but, again, prescribe as terminal conditions for the state variables K_p, K_g, G the stationary values (27):

$$\begin{array}{lllll} K_p(0) = 1, & K_g(0) = 0.02, & G(0) = 0.2, & M(0) = 2.6, & R(0) = 1, \\ K_p(t_f) = 2.2164, & K_g(t_f) = 0.53731, & G(t_f) = 0.66746. & & \end{array}$$

Our focus is on examining if emission stock remains bounded or grows unboundedly.

Figure 6 shows that for the case of low initial conditions and low atmospheric CO_2 concentration, the policy variables are able to prevent the CO_2 concentration rising unboundedly. Here too, all state and control variables move to some reasonable steady state that keeps the CO_2 concentration under control. In particular, output, consumption, and investments all remain fairly stable, except for brief initial transition associated with low initial levels of capital stocks.

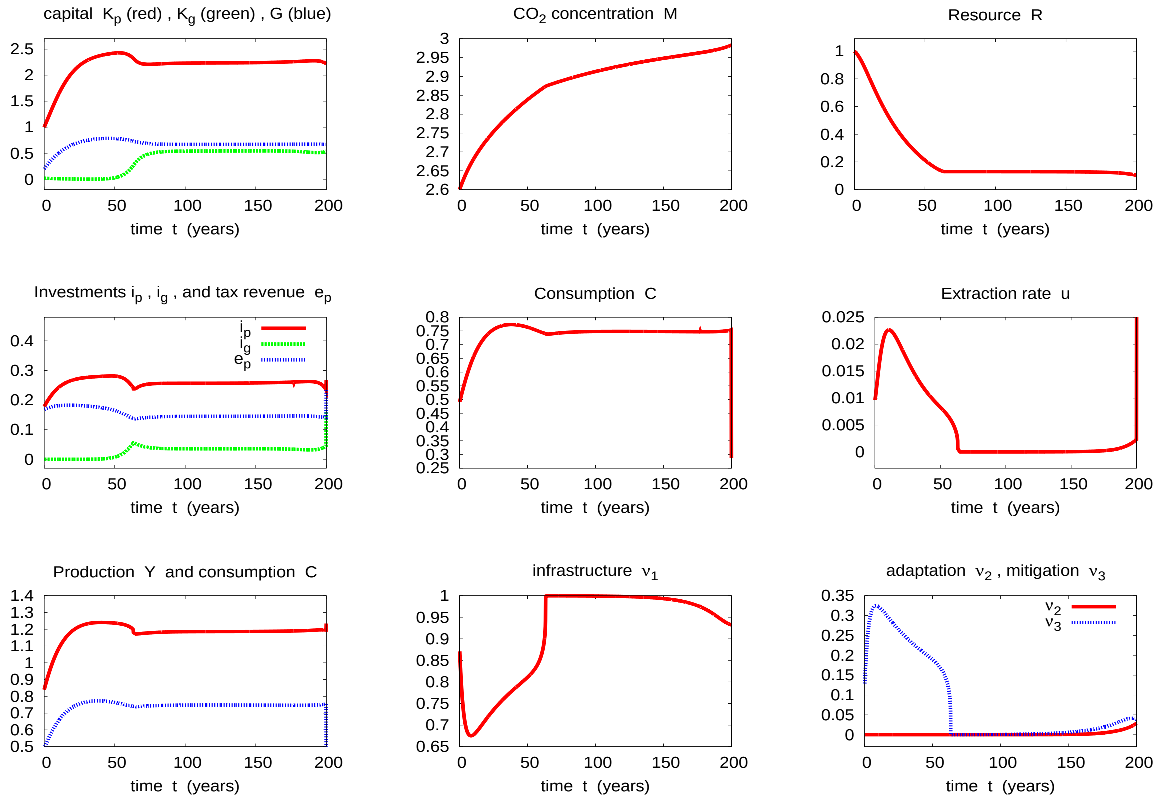

Figure 6: State and control trajectories for $T = 200$, "small" initial values $K_p(0) = 1, K_g(0) = 0.02, G(0) = 0.2, R(0) = 1, M(0) = 2.6$ and prescribed terminal stationary values $K_p(t_f) == 2.2164, K_g(t_f) = 0.53731, G(t_f) = 0.66746$ in (27). *Top row*: (left) physical capital K_p, green capital K_g and government capital G, (middle) CO_2 concentration M, (right) resource R. *Middle row*: (left) investments i_p, i_g and tax revenue e_P, (middle) consumption C (right) extraction rate u. *Bottom row*: (left) consumption C and productivity Y, (middle) infrastructure ν_1, (right) adaptation ν_2 and mitigation ν_3.

Note that, here again, the spending on traditional infrastructure is lower at the beginning which large efforts at mitigation, but, this time, no efforts at adaptation. Over time, the mitigation policy slows down. During the initial periods, the spending on mitigation starts with accounting for over 35% of the public investment in infrastructure, gradually falling to zero over the initial periods. The optimal solution, again, leaves some of the fossil fuels (R) in the ground.

5 Conclusions

As a recent study by the IPCC (2018) has shown, climate policies face great challenges in not surpassing upper limits of atmospheric CO_2 concentration. In the context of our model we can demonstrate that with proper policy actions the CO_2 can be steered down, when stocks of capital are large and the actual CO_2 concentration is above a target level. We also show that the CO_2 can be controlled not to exceed an upper limit so that emission stay bounded by a predefined upper bound. In either case climate policies can be scaled up through the use of resources allocated to the mitigation effort when climate risk, due to a lack of emission reduction, is rising or too high, and future economic, social, and ecological damages can be expected.

More generally, in all our cases, we find that the early enacting of the mitigation effort is vital for controlling the atmospheric CO_2 content whereas the adaptation policy in most cases moves up in order of importance at a later time. Infrastructure investment efforts are in most cases delayed. However, we want to note that these two types of control actions might not work, in case we are currently above or below the CO_2 target, if there are – as demonstrated in simpler models by Greiner et al (2010) and Nordhaus (2008) – tipping points and thresholds beyond which climate extremes accelerate.

We provide some dynamic estimates of how the scaling up of efforts of mitigation and adaptation can be funded and how the funds should be allocated between (traditional and climate related) infrastructure investment, mitigation and adaptation efforts. Since in this context successful mitigation policy means phasing in of renewable energy,[6] we have also explored what amount of traditional fossil energy should be left *in situ* in order to satisfy some CO_2 emission and temperature constraints.[7]. Addressing these issues required enlarging the IAM and using the solution methods for higher dimensional nonlinear control problems. We also showed that our numerical solutions for finite horizon decision model have turnpike properties similar to infinite horizon models.

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[6]See also Maurer and Semmler (2015).

[7]For a model proposing also other means of financing climate policies, for example climate bonds, see Orlov et al. (2018)

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